COMPLEX INTERPOLATION OF COUPLE $(\mathrm{X}, \mathrm{BMO})$ FOR A_1-REGULAR LATTICES

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Abstract. Recent results of A. Lerner concerning certain properties of the Fefferman-Stein maximal function are applied to show that $(\mathrm{BMO}, X)_\theta = X^\theta$, $0 < \theta < 1$, for a Banach lattice X of measurable functions on $\mathbb{R}^n$ satisfying the Fatou property such that X has order continuous norm and the Hardy-Littlewood maximal operator M is bounded in $(X^\alpha)'$ for some $0 < \alpha \leqslant 1$.

0. Introduction

Recently various classical results of harmonic analysis for important classical Banach spaces such as L_p have been generalized to their variable exponent analogues such as $\mathrm{L}_{p(\cdot)}$ and in some cases to general Banach lattices. Interpolation of such spaces has also received some attention; see, e. g., [3], [5], [11], [7]. In particular, in [7] it was established with the help of variable exponent Triebel-Lizorkin spaces that $(\mathrm{L}_{p(\cdot)}, \mathrm{BMO})_\theta = \mathrm{L}_{\frac{p(\cdot)}{1-\theta}}$ on $\mathbb{R}^n$ for $0 < \theta < 1$ along with the corresponding formula for H_1 under the assumption that the Hardy-Littlewood maximal operator M is bounded in $\mathrm{L}_{p(\cdot)}$. This extends the classical result going back to [4] saying that in the scale of complex interpolation spaces L_p one can replace the endpoint space L_∞ by BMO. In this short note we establish an extension of this result to fairly general Banach lattices. Although it appears feasible to extend the approach of [7] to this generality by studying the Triebel-Lizorkin type spaces corresponding to general Banach lattices, in this case it feels more natural to use a straightforward extension of the original argument involving application of the Fefferman-Stein maximal function, which is made possible by recent results of A. Lerner [9] extending certain properties of the Fefferman-Stein maximal function to fairly general Banach lattices. There are, of course, a number of technical difficulties to be addressed.

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1. Preliminaries

First, let us define the complex interpolation spaces. For a couple (X_0, X_1) of compatible complex Banach spaces and $0 \leqslant \theta \leqslant 1$ the complex interpolation space $(X_0, X_1)_\theta$ is defined as follows (for more detail see, e. g., [1, Chapter 4]). Let $\mathcal{F}_{X_0,X_1}$ be the space of all bounded and continuous functions $f : z \mapsto f_z$ that are defined on the strip $S = \{z \in \mathbb{C} \mid 0 \leqslant \Re z \leqslant 1\}$ and take values in $X_0 + X_1$ such that f are analytic on the interior of S, $f_{j+it} \in X_j$ for $j \in \{0, 1\}$ and all $t \in \mathbb{R}$, and $\|f_{j+it}\|_{X_j} \to 0$ as $|t| \to \infty$. The space $\mathcal{F}_{X_0,X_1}$ is equipped with the norm $\|f\|_{\mathcal{F}_{X_0,X_1}} = \sup_{t\in\mathbb{R}, j\in\{0,1\}} \|f_{j+it}\|_{X_j}$. Then space $(X_0, X_1)_\theta = \{f_\theta \mid f \in \mathcal{F}_{X_0,X_1}\}$ equipped with the norm

$$\|a\|_{(X_0,X_1)_\theta} = \inf \left\{ \|f\|_{\mathcal{F}_{X_0,X_1}} \mid f \in \mathcal{F}, f_\theta = a \right\}$$

is an interpolation space of exponent θ between X_0 and X_1. Moreover, $X_0 \cap X_1$ is dense in $(X_0, X_1)_\theta$ for $0 < \theta < 1$ (see, e. g., [1, Theorem 4.2.2]), and if $X_0 \cap X_1$ is dense in X_j for $j \in \{0, 1\}$ then $(X_0, X_1)_j = X_j$ (see, e. g., remarks after [8, Chapter 4, Theorem 1.3]).

We are now going to list some well-known standard facts about Banach lattices of measurable functions that we need in the present work; for more detail see, e. g., [6]. A Banach space X of measurable functions on a σ-finite measurable space Ω (for example, $\Omega = \mathbb{R}^n$ with the Lebesgue measure) is called a *Banach lattice* if for any $f \in X$ and a measurable function g such that $|g| \leqslant f$ almost everywhere we also have $g \in X$ and $\|g\|_X \leqslant C\|f\|_X$ with some C independent of f and g. We say that X satisfies the *Fatou property* (which is usually assumed in the literature, implicitly or otherwise) if $f_n \in X$, $\|f_n\|_X \leqslant 1$ and $f_n \to f$ almost everywhere for some f imply that $f \in X$ and $\|f\|_X \leqslant 1$. The *order dual* X' of X can be identified with the Banach lattice of measurable functions g having finite norm $\|g\|_{X'} = \sup_{f\in X, \|f\|_X \leqslant 1} \int_\Omega fg$. The Fatou property of a lattice X is equivalent to order reflexivity of X, that is to the relation $X = X''$. A Banach lattice is said to have an *order continuous norm* if $\|f_n\|_X \to 0$ for every nonincreasing sequence of functions $f_n \in X$ convegring to 0 almost everywhere. A Banach space has order continuous norm if and only if its order dual is isomorphic to the dual Banach space, i. e. $X^* = X'$. Thus, for example, $\mathrm{L}_p' = \mathrm{L}_{p'}$ for $1 \leqslant p \leqslant \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, but $\mathrm{L}_p^* = \mathrm{L}_{p'}$ holds true only for $1 \leqslant p < \infty$.

For Banach lattices X_0, X_1 and $0 < \theta < 1$ the *Calderon product* is the lattice of measurable functions f such that the norm

$$\|f\|_{X_0^{1-\theta} X_1^\theta} = \inf \left\{ \left\| |f_0|^{\frac{1}{1-\theta}} \right\|_{X_0}^{1-\theta} \left\| |f_1|^{\frac{1}{\theta}} \right\|_{X_1}^{\theta} \mid f = f_0 f_1 \right\}$$

is finite. It is well known (see, e. g., [2], [10]) that if X_0 and X_1 have the Fatou property then $X_0^{1-\theta} X_1^\theta$ is also a Banach lattice satisfying the Fatou property and its order dual can be computed as

$\left(X_0^{1-\theta}X_1^{\theta}\right)' = X_0'^{1-\theta}X_1'^{\theta}$. Setting $X_0 = \mathrm{L}_\infty$ and $X^\theta = X^\theta \mathrm{L}_\infty^{1-\theta}$ allows one to scale lattices, so that, for example, $[\mathrm{L}_p]^\theta = \mathrm{L}_{\frac{p}{\theta}}$, and we have a useful duality relation $\left(X^\theta\right)' = X'^\theta \mathrm{L}_1^{1-\theta}$. It is easy to see that if either X_0 or X_1 has order continuous norm then $X_0^{1-\theta}X_1^\theta$ also has order continuous norm. In [2] (see also [8, Chapter 4, Theorem 1.14]) it was established that Calderon products describe complex interpolation spaces between Banach lattices, i. e. $(X_0, X_1)_\theta = X_0^{1-\theta}X_1^\theta$, provided that $X_0^{1-\theta}X_1^\theta$ has order continuous norm[1].

Let X be a Banach lattice of measurable functions on Ω. The lattice $X(l^\infty)$ is the space of all measurable functions $f = \{f_j\}_{j\in\mathbb{Z}}$ on $\Omega \times \mathbb{Z}$ such that the norm $\|f\|_{X(l^\infty)} = \left\|\sup_j |f_j|\right\|_X$ is finite. This is a particular case of the general construction of a lattice with mixed norm that we will use in the present work. It is easy to see that if X satisfies the Fatou property then so does $X(l^\infty)$ and $[X(l^\infty)]^\theta = X^\theta(l^\infty)$ for all $0 < \theta < 1$. Observe that $X(l^\infty)$ never has order continuous norm. Because of this we will need the following simple proposition (which actually holds true for any lattice of measurable functions in place of l^∞; although we will only use the well-known inclusion $\subset$, we also prove the converse inclusion for completeness).

Proposition 1. *Let X_0 and X_1 be Banach lattices of measurable functions. If X_0 has order continuous norm then*

$$(X_0(l^\infty), X_1(l^\infty))_\theta = X_0^{1-\theta}X_1^\theta(l^\infty).$$

Indeed, inclusion $(X_0(l^\infty), X_1(l^\infty))_\theta \subset X_0^{1-\theta}X_1^\theta(l^\infty)$ follows at once from [2, §13.6, i], and we only need to establish the converse inclusion. Let $f = \{f_j\}_{j\in\mathbb{Z}} \in X_0^{1-\theta}X_1^\theta(l^\infty)$, and $F = \sup_j |f_j|$. Then $F \in X_0^{1-\theta}X_1^\theta = (X_0, X_1)_\theta$. This means that $F = f_\theta$ for some $f \in \mathcal{F}_{X_0,X_1}$ with an appropriate estimate on the norm. Defining $F_z = \{f_{z,j}\}_{j\in\mathbb{Z}}$ by $f_{z,j} = \frac{f_j}{F} f_z$ (with the usual convention that $\frac{0}{0} = 0$) shows that $F_z \in \mathcal{F}_{X_0(l^\infty),X_1(l^\infty)}$ with the same norm as f_z, so

$$f = F_\theta \in (X_0(l^\infty), X_1(l^\infty))_\theta$$

with an appropriate estimate on the norm. The proof of Proposition 1 is complete.

The *Hardy-Littlewood maximal operator* M is defined for all locally summable functions f by

$$Mf(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y)|dy, \quad x \in \mathbb{R}^n,$$

[1] In [7] and in some other papers it was claimed that $(X_0, X_1)_\theta = X_0^{1-\theta}X_1^\theta$ when $X_0^{1-\theta}X_1^\theta$ has the Fatou property. However, in general the Fatou property only gives $(X_0, X_1)^\theta = X_0^{1-\theta}X_1^\theta$, and a simple example of two weighted spaces $\mathrm{L}_\infty(w)$ shows that sometimes $(X_0, X_1)^\theta = X_0^{1-\theta}X_1^\theta \supsetneq (X_0, X_1)_\theta$ in this case. See, e. g., [2, §13.6].

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x with edges parallel to the coordinate axes. A locally summable nonnegative function w belongs to the *Muckenhoupt class* A_1 *with constant* c if $Mw \leqslant cw$ almost everywhere. We say that a Banach lattice X of measurable functions on $\mathbb{R}^n$ is A_1*-regular with constants* (c, m) if for any $f \in X$ there exists some majorant $w \geqslant |f|$ belonging to A_1 with constant c such that $\|w\|_X \leqslant c\|f\|_X$. By [12, Proposition 1.2] a Banach lattice X is A_1-regular if and only if M is bounded in X; thus A_1-regularity of X can justifiably be considered a rather convenient term for boundedness of M in X. The proof is very simple: an A_1-majorant for $f \in X$ gives at once the necessary estimate for Mf, and conversely an A_1-majorant can be quickly obtained from boundedness of M in X by the well-known construction due to Rubio de Francia.

With the help of the theory of Muckenhoupt weights it is rather easy to see that the A_1-regularity property is "almost self-dual" in the following sense.

Proposition 2. [12, Proposition 1.7] *Let X be a Banach lattice of measurable functions on $\mathbb{R}^n$ having either the Fatou property or order continuous norm. Suppose that X' is A_1-regular. Then X^θ is also A_1-regular for any $0 < \theta < 1$.*

The following well-known characterization of A_1 weights is very useful; in can be found in, e. g., [14, Chapter 5, §5.2].

Proposition 3. *Let w be a nonnegative locally summable function. Then $w \in \mathrm{A}_1$ with a constant c if and only if there exists an exponent $0 < q < 1$, a locally summable function f and constants $c_0, c_1 > 0$ such that $c_0 w \leqslant (Mf)^q \leqslant c_1 w$. If this holds true then constant c and constants q, c_0, c_1 can be estimated in terms of one another.*

Proposition 3 is a consequence of the reverse Hölder inequality satisfied by A_1 weights. It allows a very easy proof of the following result.

Proposition 4. *Let X be an A_1-regular Banach lattice of measurable functions on $\mathbb{R}^n$. Then lattices X^θ and $X^{1-\theta}\mathrm{L}_1^\theta$ are also A_1-regular for all $0 < \theta < 1$.*

Indeed, A_1-regularity of X^θ is a trivial corollary to Proposition 3, and it is otherwise established at once using the Hölder inequality. More generally, the Hölder inequality shows that for any two A_1-regular lattices A and B lattice $A^{1-\theta}B^\theta$ is also A_1-regular (see, e. g., [12, Proposition 3.4]), and this also implies A_1-regularity of X^θ since lattice L_∞ is trivially A_1-regular. It is, however, well known that lattice L_1 is not A_1-regular, so A_1-regularity of $X^{1-\theta}\mathrm{L}_1^\theta$ is a bit more tricky. Suppose that $f \in X^{1-\theta}\mathrm{L}_1^\theta$. We may assume that $f \geqslant 0$ and $\|f\|_{X^{1-\theta}\mathrm{L}_1^\theta} = 1$. Then $f = g^{1-\theta}h^\theta$ with some $g \in X$ and $h \in \mathrm{L}_1$ with $\|g\|_X \leqslant 2$ and $\|h\|_{\mathrm{L}_1} \leqslant 2$. Let w be an A_1-majorant for g in X. Then by Proposition 3 weight w is pointwise equivalent to $(Ma)^q$ almost everywhere

with some locally summable function a and with $0 < q < 1$ depending only on the A_1-regularity constants of X. Since M is bounded in L_p for any $1 < p \leqslant \infty$ we have an estimate $\left\| (M[h^\alpha])^{\frac{1}{\alpha}} \right\|_{\mathrm{L}_1} \leqslant c$ with some c independent of f for any $0 < \alpha < 1$. Observe that f is dominated by $u = c_1 (Ma)^{q(1-\theta)} (Mh^\alpha)^{\frac{1}{\alpha}\theta}$ and $\|u\|_{X^{1-\theta}\mathrm{L}_1^\theta} \leqslant c_2$ with some c_1 and c_2 independent of f. We claim that with a certain choice of α we have $u \in \mathrm{A}_1$ with a constant independent of f. Indeed, by the Hölder inequality

$$(1) \qquad \frac{1}{c_1|Q|} \int_Q u \leqslant \left(\frac{1}{|Q|} \int_Q (Ma)^{pq(1-\theta)} \right)^{\frac{1}{p}} \left(\frac{1}{|Q|} \int_Q (Mh^\alpha)^{\frac{1}{\alpha} p' \theta} \right)^{\frac{1}{p'}}$$

for any cube $Q \subset \mathbb{R}^n$ and $1 < p < \infty$. If we choose the parameters so that

$$(2) \qquad pq(1-\theta) < 1, \quad \frac{p'}{\alpha}\theta < 1,$$

then $(Ma)^{pq(1-\theta)} \in \mathrm{A}_1$ and $(Mh^\alpha)^{\frac{1}{\alpha}p'\theta} \in \mathrm{A}_1$ by Proposition 3, and therefore (1) and (2) imply that

$$\frac{1}{c_1|Q|} \int_Q u \leqslant c_2 (Ma)^{q(1-\theta)} (Mh^\alpha)^{\frac{1}{\alpha}\theta} = c_2 u$$

almost everywhere with some constant c_2, i. e. $u \in \mathrm{A}_1$ with an appropriate estimate on the constant. Rewriting (2) as $\frac{\alpha}{\alpha-\theta} < p < \frac{1}{q(1-\theta)}$ we see that we can always choose an appropriate p if we take any $1 > \alpha > \frac{\theta}{1-q(1-\theta)}$. The proof of Proposition 4 is complete.

Let f be a measurable function on $\mathbb{R}^n$. The *nonincreasing rearrangement* f^* of f is defined by

$$f^*(t) = \inf \{ \lambda > 0 \mid |\{ x \in \mathbb{R}^n \mid |f(x)| > \lambda \}| \leqslant t \}, \quad 0 < t < \infty.$$

Let S_0 be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$f^*(+\infty) = \lim_{t\to\infty} f^*(t) = 0.$$

It is easy to see that S_0 contains all measurable functions supported on sets of finite measure and also $\mathrm{L}_p \subset S_0$ for all $0 < p < \infty$. Thus if X is a Banach lattice of measurable functions having order continuous norm then $S_0 \cap X$ is dense in X. Density of $S_0 \cap X$ in a lattice X is a somewhat more general assumption than density of simple functions with compact support in X; for example, simple functions with compact support are not dense in a lattice $\mathrm{L}_\infty(w) = w\mathrm{L}_\infty$ with weight $w(x) = (1+|x|)^{-1}$ but at the same time we have $\mathrm{L}_\infty(w) \subset S_0$.

Now we will briefly discuss some of the results involving the Fefferman-Stein sharp maximal function. The Fefferman-Stein maximal function $f^\sharp$ on $\mathbb{R}^n$ is defined for a locally integrable function f by

$$f^\sharp(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x with edges parallel to the coordinate axes and and $f_Q = \frac{1}{|Q|} \int_Q f(z) dz$ is the average value of f over Q with respect to the Lebesgue measure. Space BMO can then be defined as the space of all locally integrable functions f such that $f^\sharp \in \mathrm{L}_\infty$ modulo constants equipped with the norm $\|f\|_{\mathrm{BMO}} = \|f^\sharp\|_{\mathrm{L}_\infty}$ that turns BMO into a Banach space; for more detail see, e. g., [14, Chapter 4]. We have continuous inclusion $\mathrm{L}_\infty \subset \mathrm{BMO}$, but L_∞ is a proper subspace of BMO. The usefulness of BMO in harmonic analysis stems mainly from the fact that BMO is dual to the Hardy space H_1 and many operators of interest are not bounded in L_∞ but act boundedly from L_∞ to BMO if suitably defined on this space.

Theorem 5 ([9, Corollary 4.3]). *Suppose that X is an A_1-regular real Banach lattice of measurable functions on $\mathbb{R}^n$ having the Fatou property. Then the following conditions are equivalent.*

(1) *X' is A_1-regular.*
(2) *There exists some $c > 0$ such that $\|f\|_X \leqslant c\|f^\sharp\|_X$ for all $f \in S_0 \cap X$.*

This theorem can be considered an extension of well-known classical results for $X = \mathrm{L}_p$ (see, e. g., [14]). The proof involves a certain linearization of M, pointwise equivalence of $f^\sharp$ and $MM^\sharp_\lambda$ for some λ (where $M^\sharp_\lambda$ denotes the Strömberg local sharp maximal function) and the fact that $M^\sharp_\lambda$ is dual to M in the sense that $\int |fg| \leqslant c \int M^\sharp_\lambda f \, Mg$ for suitable f and g.

It is easy to see that the estimate in Theorem 5 can be extended to the entire lattice X provided that $S_0 \cap X$ is dense in X, and the complex lattices can be included as well.

Proposition 6. *Suppose that X is a Banach lattice of measurable functions on $\mathbb{R}^n$ having the Fatou property, both X and X' are A_1-regular and $S_0 \cap X$ is dense in X. Then there exists some $c > 0$ such that*

$$\|f\|_X \leqslant c\|f^\sharp\|_X \tag{3}$$

for all $f \in X$.

Indeed, suppose that $f \in X$ under the conditions of Proposition 6, f is real, and let $f_n \in S_0 \cap X$ be a sequence such that $f_n \to f$ in X. Observe that the Fefferman-Stein maximal function is subadditive and $g^\sharp \leqslant 2Mg$ for all locally summable functions g. Therefore Theorem 5 allows us to carry out the estimate

$$\begin{aligned} \frac{1}{c}\|f_n\|_X \leqslant \|f_n^\sharp\|_X \leqslant \|f^\sharp\|_X + \|(f - f_n)^\sharp\|_X \leqslant \\ \|f^\sharp\|_X + 2\|M(f - f_n)\|_X \leqslant \|f^\sharp\|_X + 2\|M\|_{X \to X}\|f - f_n\|_X. \end{aligned} \tag{4}$$

Passing to the limit $n \to \infty$ in (4) yields (3) for all real functions $f \in X$. If $f \in X$ is complex then (3) implies that $\|\Re f\|_X \leqslant c\|(\Re f)^\sharp\|_X \leqslant c\|f^\sharp\|_X$ because $(\Re f)^\sharp \leqslant f^\sharp$ almost everywhere, and the same estimate holds true for $\Im f$. Combining these estimates together yields

$$\|f\|_X \leqslant \|\Re f\|_X + \|\Im f\|_X \leqslant 2c\|f^\sharp\|_X.$$

It is easy to see that $\mathrm{L}_\infty^\alpha \subset \mathrm{L}_\infty$ for all $0 < \alpha \leqslant 1$. It is also not hard to verify that BMO (which is also a lattice) satisfies the same property.

Proposition 7. *Suppose that $f \in \mathrm{BMO}$ and $f \geqslant 0$. Then $f^\alpha \in \mathrm{BMO}$ for all $0 < \alpha \leqslant 1$.*

Since $f^\alpha - f^\alpha \vee 1$ is a bounded function under the conditions of Proposition 7 (and hence $f^\alpha - f^\alpha \vee 1 \in \mathrm{BMO}$), it suffices to verify that $f^\alpha \vee 1 = (f \vee 1)^\alpha \in \mathrm{BMO}$. We have $g = f \vee 1 \in \mathrm{BMO}$ because BMO is a lattice, and then $g^\alpha \in \mathrm{BMO}$ is clear because the map $F : y \mapsto y^\alpha$ is contractive for $y \geqslant 1$ and therefore oscillations of g^α do not increase compared to the corresponding oscillations of g. Perhaps the easiest way to verify this formally is via the Strömberg characterization of BMO mentioned above (see, e. g., [14, Chapter 4, §6.6]; this also involves the local sharp maximal function $M_\lambda^\sharp$ used in the proof of Theorem 5) which states that $g \in \mathrm{BMO}$ if and only if there exist some constants $0 < \gamma < \frac{1}{2}$ and $\lambda > 0$ such that

$$\inf_{c_Q \in \mathbb{R}} |\{x \in Q \mid |g(x) - c_Q| > \lambda\}| \leqslant \gamma |Q| \tag{5}$$

for any cube $Q \subset \mathbb{R}^n$. It is easy to see that if F is a contractive map then (5) implies

$$\inf_{c_Q \in \mathbb{R}} |\{x \in Q \mid |F \circ g(x) - F(c_Q)| > \lambda\}| \leqslant \gamma |Q|$$

for any cube $Q \subset \mathbb{R}^n$, so $F \circ g \in \mathrm{BMO}$ if $g \in \mathrm{BMO}$.

Proposition 8. *Let X be a Banach lattice and suppose that $X^\alpha \cap \mathrm{BMO}$ is a subset of $X^{\theta\alpha}$ for some $0 < \alpha, \theta < 1$. Then $X \cap \mathrm{BMO}$ is a subspace of X^θ for all $0 < \eta < 1$.*

Indeed, since BMO is a lattice, it is sufficient verify the inclusion $X \cap \mathrm{BMO} \subset X^\eta$ for nonnegative functions. Suppose that $f \in X \cap \mathrm{BMO}$ and $f \geqslant 0$ almost everywhere. Then $f^\alpha \in X^\alpha$ and by Proposition 7 we have $f^\alpha \in \mathrm{BMO}$. Thus $f^\alpha \in X^\alpha \cap \mathrm{BMO} \subset X^{\theta\alpha}$ and therefore $f \in X^\theta$.

2. Interpolation

We are now ready to state the main result.

Theorem 9. *Suppose that X is a Banach lattice of measurable functions on $\mathbb{R}^n$ having the Fatou property and order continuous norm, and lattice $(X^\alpha)'$ is A_1-regular for some $0 < \alpha \leqslant 1$. Then*

$$(\mathrm{BMO}, X)_\theta = X^\theta \tag{6}$$

for any $0 < \theta < 1$.

We will give a few remarks before passing to the proof of Theorem 9. The assumption that $(X^\alpha)'$ is A_1-regular combined with the assumption that X has order continuous norm cannot be dropped from Theorem 9. Otherwise we would have had

$$(\mathrm{BMO}_b, \mathrm{L}_\infty)_\theta = (\mathrm{BMO}, \mathrm{L}_\infty)_\theta = \mathrm{L}_\infty, \tag{7}$$

where BMO_b is the closure of L_∞ in BMO. Equation (7) implies that $\mathrm{BMO}_b = \mathrm{L}_\infty$ by [13, Theorem 1.7]. However, it is well known that

$$\mathrm{L}_\infty \neq \mathrm{VMO} \subset \mathrm{BMO}_b;$$

see, e. g., [14, Chapter 4, §6.8]. On the other hand, it seems that A_1-regularity of $(X^\alpha)'$ should imply order continuity of the norm of X. This is true at least in the case of variable exponent Lebesgue spaces $X = \mathrm{L}_{p(\cdot)}$ since $(X^\alpha)' = \mathrm{L}_{(p(\cdot)/\alpha)'}$ and $\operatorname{ess\,sup} p(\cdot) = \infty$ would imply $\operatorname{ess\,inf} (p(\cdot)/\alpha)' = 1$ which contradicts A_1-regularity of $(X^\alpha)'$ by [3, Theorem 4.7.1].

It is easy to see that if the conditions of Theorem 9 are satisfied for some α then they are satisfied for all smaller values of α, and the lattice X^β is A_1-regular for all $0 < \beta < \alpha$. Indeed, under the conditions of Theorem 9 lattice $X^\beta = (X^\alpha)^{\frac{\beta}{\alpha}}$ is A_1-regular for any $0 < \beta < \alpha$ by Proposition 2, and lattice

$$(X^\beta)' = (X')^\beta \mathrm{L}_1^{1-\beta} = \left[(X')^\alpha \mathrm{L}_1^{1-\alpha}\right]^{\frac{\beta}{\alpha}} \mathrm{L}_1^{1-\frac{\beta}{\alpha}} = [(X^\alpha)']^{\frac{\beta}{\alpha}} \mathrm{L}_1^{1-\frac{\beta}{\alpha}}$$

is A_1-regular for the same values of β by Proposition 4.

We now provide a couple of applications for Theorem 9. Muckenhoupt weights $w \in \mathrm{A}_p$ for $1 < p < \infty$ are exactly those for which the Hardy-Littlewood maximal operator M is bounded in the weighted Lebesgue space $\mathrm{L}_p(w)$ with norm defined by

$$\|f\|_{\mathrm{L}_p(w)}^p = \int |f|^p w$$

(here we use this classical definition for the sake of simplicity; in [12], for example, the same space was denoted by $\mathrm{L}_p\left(w^{-\frac{1}{p}}\right)$ which gives more consistency with the endpoint $p = \infty$ and Calderon products). We can naturally extend this definition to $p = \infty$ by $\mathrm{A}_\infty = \bigcup_{p>1} \mathrm{A}_p$; for more detail on Muckenhoupt weights see, e. g., [14, Chapter 5].

Corollary 10. *Suppose that* $w \in \mathrm{A}_\infty$. *Then for any* $0 < \theta < 1$ *we have*

$$(\mathrm{BMO}, \mathrm{L}_1(w))_\theta = \mathrm{L}_{\frac{1}{\theta}}(w). \tag{8}$$

We want to verify that the conditions of Theorem 9 are satisfied for $X = \mathrm{L}_1(w)$ under the conditions of Corollary 10. Indeed,

$$(\mathrm{L}_1(w)^\alpha)' = (\mathrm{L}_{\frac{1}{\alpha}}(w))' = \mathrm{L}_{\frac{1}{1-\alpha}}\left(w^{-1}\right),$$

and A_1-regularity of this lattice for suitable values of α follows from the following simple proposition.

Lemma 11. *Suppose that $w \in A_\infty$. Then $w^{-1} \in A_\infty$.*

There are many straightforward ways to establish Lemma 11 using numerous characterizations of the A_∞ weights; here we are going to use nothing more than Proposition 2. Indeed, by the assumptions we have $w \in A_{p_0}$ with some $1 < p_0 < \infty$. Then lattice $[L_{p'}(w^{-1})]' = L_p(w)$ is A_1-regular for all $p \geqslant p_0$, and by Proposition 2 lattice

$$\left[L_{p'}\left(w^{-1}\right)\right]^{\frac{1}{q}} = L_{p'q}\left(w^{-1}\right)$$

is A_1-regular for all $p \geqslant p_0$ and $q > 1$, so $w^{-1} \in A_{p'q} \subset A_\infty$ as claimed.

Application of Theorem 9 to the case $X = L_{p(\cdot)}$ yields part of the results from [7]; for definitions and general discussion of variable exponent Lebesgue spaces $L_{p(\cdot)}$ see, e. g., [3].

Corollary 12. *Let $p(\cdot) : \mathbb{R}^n \to [1, \infty]$ be a measurable function such that $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty$ and suppose that $L_{p(\cdot)}$ is A_1-regular. Then*

$$(\mathrm{BMO}, L_{p(\cdot)})_\theta = L_{\frac{p(\cdot)}{\theta}} \tag{9}$$

for all $0 < \theta < 1$.

Variable exponent Lebesgue spaces $L_{p(\cdot)}$ can be regarded as a natural generalization of the standard Lebesgue spaces L_p, which correspond to the case $p(\cdot) = p$, and lattice operations in spaces $L_{p(\cdot)}$ behave largely the same as their Lebesgue space counterparts. Observe that A_1-regularity of $L_{p(\cdot)}$ implies by [3, Theorem 4.7.1] that $\operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x) > 1$, and by [3, Theorem 5.7.2] it follows that lattice $\left[L_{p(\cdot)}\right]' = L_{p'(\cdot)}$ is also A_1-regular. Condition $\operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty$ easily implies (see, e. g., [3, Lemma 2.3.16]) that $L_{p(\cdot)}$ has order continuous norm. Therefore we can apply Theorem 9 with $\alpha = 1$, which concludes the proof of Corollary 12.

We now begin the proof of Theorem 9. First, observe that T. Wolff's well-known result concerning gluing of interpolation scales allows us to reduce it to the case $\alpha = 1$.

Theorem 13 ([15, Theorem 2])**.** *Let A_1, A_2, A_3, A_4 be Banach spaces. Suppose that $A_1 \cap A_4$ is a dense subspace of A_2 and of A_3 and*

$$A_3 = (A_2, A_4)_\gamma, \quad A_2 = (A_1, A_3)_\delta$$

with some $0 < \gamma, \delta < 1$. Then

$$A_2 = (A_1, A_4)_\xi, \quad A_3 = (A_1, A_4)_\psi$$

for $\xi = \frac{\gamma\delta}{1-\delta+\gamma\delta}$ and $\psi = \frac{\gamma}{1-\delta+\gamma\delta}$.

Indeed, suppose that under the conditions of Theorem 9 we have established that

$$(\mathrm{BMO}, X^\alpha)_\eta = X^{\eta\alpha} \tag{10}$$

for all $0 < \eta < 1$. First, suppose that $\theta < \alpha$ and let $A_1 = \mathrm{BMO}$, $A_2 = X^\theta$, $A_3 = X^\alpha$ and $A_4 = X$. Equation (10) implies that $\mathrm{BMO} \cap X^\alpha$ is a subspace of $X^{\eta\alpha}$, so $A_1 \cap A_4$ is a subspace of A_2 and A_3 by Proposition 8. The density assumptions of Theorem 13 are satisfied because $\mathrm{BMO} \cap X \supset \mathrm{L}_\infty \cap X$, which is a dense subspace of $(\mathrm{L}_\infty, X)_\zeta = X^\zeta$ for all $0 < \zeta < 1$. The conditions of Theorem 13 are satisfied with values $\delta = \frac{\theta}{\alpha}$ and $\gamma = \frac{\alpha - \theta}{1 - \theta}$, and thus $X^\theta = A_2 = (A_1, A_4)_\xi = (\mathrm{BMO}, X)_\theta$ ($\xi = \theta$ follows from an easy computation), i. e. (6) is satisfied for all $0 < \theta < \alpha$; we also get $X^\alpha = A_3 = (A_1, A_4)_\psi = (\mathrm{BMO}, X)_\alpha$, which is (6) for $\theta = \alpha$. The remaining case $\alpha < \theta < 1$ is then easily established by the reiteration theorem (see, e. g., [1, Theorem 4.6.1]): we have

$$X^\theta = (X^\alpha, X)_\eta = ((\mathrm{BMO}, X)_\alpha, (\mathrm{BMO}, X)_1)_\eta = (\mathrm{BMO}, X)_{(1-\eta)\alpha + \eta} = (\mathrm{BMO}, X)_\theta$$

for $\eta = \frac{\theta - \alpha}{1 - \alpha}$.

Thus we only need to verify (10) for all sufficiently small α under the conditions of Theorem 9. Since we can always make α smaller, we may assume that lattices X^β and $(X^\beta)'$ are A_1-regular for all $0 < \beta \leqslant \alpha$. For convenience we replace X^α by X; thus lattices X^β and $(X^\beta)'$ are A_1-regular for $0 < \beta \leqslant 1$, and we need to verify that $(\mathrm{BMO}, X)_\eta = X^\eta$ for all $0 < \eta < 1$. The proof now follows the standard pattern. Let $0 < \theta < 1$. Since $\mathrm{L}_\infty \subset \mathrm{BMO}$, we have $(\mathrm{BMO}, X)_\theta \supset (\mathrm{L}_\infty, X)_\theta = X^\theta$, and only the converse inclusion needs to be established. Because $\mathrm{BMO} \cap X$ is dense in $(\mathrm{BMO}, X)_\theta$, it suffices to verify this inclusion on $\mathrm{BMO} \cap X$. Suppose that $a \in (\mathrm{BMO}, X)_\theta \cap (\mathrm{BMO} \cap X)$; then $a = f_\theta$ with some $f \in \mathcal{F}_{\mathrm{BMO}, X}$ with $\|f\|_{\mathcal{F}_{\mathrm{BMO},X}} \leqslant 2\|a\|_{(\mathrm{BMO},X)_\theta}$. We enumerate all cubes $\{Q_j\}_{j \in \mathbb{N}}$ containing 0 and having rational coordinates of the vertices and define a function $g = \{g_j\}_{j \in \mathbb{N}}$ on the strip S by

$$g_{z,j}(x) = \frac{1}{|Q_j|} \int_{Q_j + x} \left(f_z - \frac{1}{|Q_j|} \int_{Q_j + x} f_z \right) \frac{\overline{f_\theta - \frac{1}{|Q_j|} \int_{Q_j + x} f_\theta}}{\left| f_\theta - \frac{1}{|Q_j|} \int_{Q_j + x} f_\theta \right|}$$

for all $z \in S$, $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$. It is easy to see that g is continuous on the strip S and analytic in the interior of S. Moreover, we have estimates

$$\sup_j |g_{it,j}(x)| \leqslant \sup_j \frac{1}{|Q_j|} \int_{Q_j + x} \left| f_{it}(x) - \frac{1}{|Q_j|} \int_{Q_j + x} f_{it}(x) \right| \leqslant f^\sharp_{it}(x) \leqslant \|f_{it}\|_{\mathrm{BMO}} \leqslant \|f\|_{\mathcal{F}_{\mathrm{BMO},X}}$$

and

$$\sup_j |g_{1+it,j}(x)| \leqslant 2 M f_{1+it}(x)$$

for all $t \in \mathbb{R}$ and almost all $x \in \mathbb{R}^n$, so $\|g_{it}\|_{\mathrm{L}_\infty(l^\infty)} \leqslant \|f\|_{\mathcal{F}_{\mathrm{BMO},X}}$ and $\|g_{1+it}\|_{X(l^\infty)} \leqslant 2\|M f_{1+it}\|_X \leqslant c_1 \|f_{1+it}\|_X \leqslant c_1 \|f\|_{\mathcal{F}_{\mathrm{BMO},X}}$ for all $t \in \mathbb{R}$

with some constant $c_1 > 1$ independent of a. These estimates also imply that $\|g_{it}\|_{L_\infty(l^\infty)} \to 0$ and $\|g_{1+it}\|_{X(l^\infty)} \to 0$ as $t \to \infty$. Thus $g \in \mathcal{F}_{L_\infty(l^\infty),X(l^\infty)}$ and $\|g\|_{\mathcal{F}_{L_\infty(l^\infty),X(l^\infty)}} \leqslant c_1\|f\|_{\mathcal{F}_{\mathrm{BMO},X}} \leqslant 2c_1\|a\|_{(\mathrm{BMO},X)_\theta}$. Therefore $g_\theta \in (L_\infty(l^\infty), X(l^\infty))_\theta = X^\theta(l^\infty)$ by Proposition 1 with

$$\|g_\theta\|_{X^\theta(l^\infty)} \leqslant 2c_1\|a\|_{(\mathrm{BMO},X)_\theta}. \tag{11}$$

Observe that by Proposition 4 applied to A_1-regular lattices X and X' lattices X^θ and $(X^\theta)' = X'^\theta L_1^{1-\theta}$ are also A_1-regular, so by Proposition 6 we have the estimate

$$\|a\|_{X^\theta} \leqslant c\|a^\sharp\|_{X^\theta} \tag{12}$$

for all $a \in X^\theta$ with some c independent of a. Since the function under the supremum in the definition of the Fefferman-Stein maximal function depends continuously on the coordinates of the vertices of the cube Q, the maximal function $f_\theta^\sharp$ takes the same values if we only take cubes with rational coordinates of the vertices. Therefore

$$a^\sharp(x) = f_\theta^\sharp(x) = \sup_j \frac{1}{|Q_j|}\int_{x+Q_j}\left| f_\theta - \frac{1}{|Q|}\int_{x+Q_j} f_\theta \right| = \sup_j g_{\theta,j}(x)$$

for all $x \in \mathbb{R}^n$, which links (11) and (12) together:

$$\|a\|_{X^\theta} = \|f_\theta\|_{X^\theta} \leqslant c\|f_\theta^\sharp\|_{X^\theta} = c\|g_\theta\|_{X^\theta(l^\infty)} \leqslant 2c_1\|a\|_{(\mathrm{BMO},X)_\theta}.$$

Thus we have verified the claimed continuous inclusion $(\mathrm{BMO}, X)_\theta \subset X^\theta$. The proof of Theorem 9 is complete.

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